NODEs
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Introduction

Numerical Methods

Eulers Method:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Convergence of Euler's Method

$$\left|\frac{\partial f}{\partial y}\right| \leq L$$

$$L \leq \max_{(t,y)\in R}\left|\frac{\partial f}{\partial y}\right|$$

$$\left|y''(t)\right| \leq M, \quad a \leq t \leq b$$

$$D = e^{(b-a)L}\frac{M}{2L}$$

and with $e_n = y_n - y(t_n)$ we have the global error is bounded by Dh in magnitude: $|e_n| \leq Dh$,for $n = 0, 1, ..., N$.

The Flow Map

Flow Map: fixing t_0 and h we may consider the map from y_0 to $y(t_0 + h; t_o, y_0)$. This is the flow map, written:

$$y(t_0 + h; t_0, y_0) = \Phi_{t_0,h}(y_0)$$

Which is actually a family of maps.
Flow Map Approximation: can be viewing of the form:

$$\hat{\Phi}_{t,h}(y) = y + hf(t, y)$$

Methods which approximate the solution through iteration of an approximate flow map are called one-step methods.

Taylor Series Methods

$$y' = f(t, y)$$

$$y'' = f_y y' + f_t = f_y f + f_t$$

Thus

$$\Phi_{t,h}(y) = y + hf(t, y) + \frac{1}{2}h^2\left(f_y(t, y)f(t, y) + f_t(t, y)\right) + \frac{1}{6}y'''h^3 + \dots$$

Convergence of One-Step Methods

Polynomial Interpolation

The Lagrange interpolating polynomials $\ell_i, i = 1, \dots, s$ for a set of abscissae are defined by:

$$\ell_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{s}\frac{x - c_j}{c_i - c_j}$$

Defining the weights:

$$b_i = \int_0^1 \ell_i(x)\mathrm{d}x$$

The quadrature formula becomes:

$$\int_0^1 g(x)\mathrm{d}x \approx \int_0^1 P(x)\mathrm{d}x = \sum_{i=1}^{s} b_i g(c_i)$$

Runge-Kutta Methods

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, \dots, s$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i)$$

Here, s is the number of stages of the RK method, b_i are the weights and a_{ij} are the internal coefficients.
Butcher Tables:

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Order Conditions

For a method to have order $p = 1$ we need:

$$\sum_{i=1}^{s} b_i = 1$$

For a method to have order $p = 2$, it must satisfy order $p = 1$ and:

$$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$$

For a method to have order $p = 3$ it must satisfy $p = 2$ and:

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}, \quad \sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$$

Equilibrium points

Equilibrium Points: is a point of $\frac{dy}{dt} = f(y)$ for which $f(y^*) = 0$.
An equilibrium point of the ODE corresponds to a fixed point of the flow map. A point $y^* \in \mathbb{R}^d$ such that

$$\phi_t(y^*) = y^*, \forall t > 0$$

Asymptotically Stable: if

- y^* is stable
- Solutions started sufficiently near to y^* tend to y^* as $t \to \infty$

Theorem 4.3.2: Suppose $f(y) = \frac{dy}{dt}$ is C^2 and has an equilibrium point y^*. If the eigenvalues of $J = f'(y^*)$ all lie strictly in the left complex half-plane, then the equilibirum point y^* is asymptotically stable. If J has any e-val in the right complex half plane then y^* is an unstable point.
Dahlquist Test Equation: $y' = \lambda y, \quad \lambda \in \mathbb{C}$

- Has complex valued solution $y(t) = e^{\lambda t}y_0$
- Equilibrium point $y^* = 0$
- Asymptotically stable if $\mathrm{Re}(\lambda) < 0$

Stability Function: $y_{n+1} = R(h\lambda)y_n$
$R(\mu) = 1 + \mu b^T(I - \mu A)^{-1}\mathbf{1}$

- **Euler:** $R(h\lambda) = 1 + h\lambda$
  The fixed point $y^* = 0$ is asymptotically stable if when we start near zero, we tend to it, i.e
  $|1 + h\lambda| < 1$
- **Implicit Euler:** $R(h\lambda) = (1 - h\lambda)^{-1}$
- **Trapezium:** $R(h\lambda) = \frac{1+h\lambda/2}{1-h\lambda/2}$
- **Implicit Midpoint:** $R(h\lambda) = \frac{1+h\lambda/2}{1-h\lambda/2}$

Determine $h\lambda$ such that $|R(h\lambda)| \leq 1$, this is called **the region of absolute stability** of the numerical method.

Fixed Points

Set of Fixed Points: $\mathcal{F} = \{y \in \mathbb{R}^d : f(y) = 0\}$
For a numerical map Ψ_h, the fixed point may depend on h as well as f. We denote the set of fixed points Ψ_h by $\mathcal{F}_h = \{y \in \mathbb{R}^d : \Psi_h(y) = y\}$
A-stable: If stability region includes entire left half-plane
L-stable: If A-stable and $R(\mu) \to 0$ as $\mu \to \infty$
Thm: Given an RK method with stability function R, then the method is A-stable iff:

- All poles of R lie strictly in the right half plane
- $|R(it)| \le 1$ for all $t \in \mathbb{R}$

Linear Multistep Methods

Definition 5.0.1: A linear k-step method is defined as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f\left(y_{n+j}\right)$$

Where $\alpha_k \neq 0$ and either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$. Usually the coefficients are normalized such that either $\alpha_k = 1$ or $\sum_j \beta_j = 1$.
Order of Accuracy:

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \quad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$$

The θ-method: generalises all linear one-step methods:

$$y_{n+1} - y_n = h(1-\theta) f(y_n) + h\theta f(y_{n+1})$$

Residual: The residual of a linear multistep method at time t_{n+k}:

$$r_n := \sum_{j=0}^{k} \alpha_j y\left(t_{n+j}\right) - h \sum_{j=0}^{k} \beta_j y'\left(t_{n+j}\right)$$

Consistency: Equivalent conditions for a linear multistep method to have order of consistency p are:

- The coefficients α_j and β_j satisfy

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} \alpha_j j^i = i \sum_{j=0}^{k} \beta_j j^{i-1}$$

  for $\quad i = 1, \ldots, p$

- The polynomials $\rho(\zeta)$ and $\sigma(\zeta)$ satisfy:

$$\rho\left(e^z\right) - z\sigma\left(e^z\right) = \mathcal{O}\left(z^{p+1}\right)$$

- The polynomials $\rho(\zeta)$ and $\sigma(\zeta)$ satisfy:

$$\frac{\rho(z)}{\log z} - \sigma(z) = \mathcal{O}\left((z-1)^p\right)$$

Root Condition and Zero-Stability

Root Condition: A linear k-step method is said to satisfy the root condition if all roots ζ of $\rho(\zeta) = 0$ lie in the unit disc ($|\zeta| \le 1$) and any root of modulus one has multiplicity one.
Theorem 5.4.1: A multistep method is convergent iff the order is $p \ge 1$ and satisfies the root condition.

Geometric Integration

Definitions

- **Abscissa:** The distance from a point to the vertical or y -axis, measured parallel to the horizontal or x -axis; the x -coordinate.
- **Implicit:** The method is defined implicitly by an equation that has to be solved to advance the step (e.g. contains a function that depends on y_{n+1}.
- **Explicit:** Calculates the state of a system at a later time from a the state of the system at a current time.
- **Consistency:** A method is consistent of order p if, in a single timestep, the difference between the exact and approximate solutions is $O(h^{p+1})$.
- **Stability:** A method is stable if the difference between numerical solutions grows by a bounded amount as h tends to zero.
- **Local Error:** error introduced in one step of a numerical method.
- **Global Error:** $ge = |y_n - y(t_n)|$.
- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
- **First Order Method:** When Euler's method is applied with fixed stepsize on a finite time interval, the norm of the global error is bounded by a constant times the stepsize.
- Local error for Euler's method satisfies:
  $\mathrm{le}_n \le |y''(t_n)| h^2/2$
- **Liouvilles Theorem:** Flow maps of divergence free systems are volume preserving.